

# Supercritical general branching processes conditioned on extinction are subcritical

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## Abstract

It is well known that a simple, supercritical Bienaymé-Galton-Watson process turns into a subcritical such process, if conditioned to die out. We prove that the corresponding holds true for general, multi-type branching, where child-bearing may occur at different ages, life span may depend upon reproduction, and the whole course of events is thus affected by conditioning upon extinction.

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## 1 Introduction

The theory of branching processes was born out of Galton's famous family extinction problem. Later, interest turned to populations not dying out and their growth and stabilisation. In more recent years, extinction has retaken a place in the foreground, for reasons from both conservation and evolutionary biology. The time and path to extinction of subcritical general populations was studied in [5]. Here, time structure is crucial, and life spans and varying bearing ages cannot be condensed into simple, generation counting Bienaymé-Galton-Watson processes. Thus, the question arises whether (non-critical) general branching populations bound for extinction must behave like subcritical populations.

We answer this in the affirmative: a general, multitype branching process conditioned to die out, remains a branching process, but one almost surely dying out. If the original process was supercritical but with a strictly positive risk of extinction, the new process is subcritical.

Formulated in such a loose manner, this fact belongs to the folklore of branching, but actually it has been proved only for Bienaymé-Galton-Watson processes, [1], p. 52. A moment's afterthought tells us that it remains true for age-dependent branching processes of the Bellman-Harris type, where individuals have i.i.d. life spans, and split into i.i.d. random numbers of children,

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independently of life span, the time structure thus not being affected by the conditioning.

But what if the flow of time is no longer independent of reproduction? Even the simplest case, that of a splitting reproduction at death, but not independently of age at death/splitting, would seem to offer difficulties, and the same certainly applies to the more realistic general processes where reproduction occurs as a point process during life, thus mimicking the yearly births of wildlife, or the even more erratic reproduction pattern of humans.

The conceptual framework is intuitive. Starting from an Eve, individuals live and give birth independently of one another. At birth each individual inherits a *type* from her mother. The type, in its turn determines the probability measure over all possible life careers, including a life span and a marked point process which reports the successive ages at bearing, and the types of children at the various bearings. Note that multiple births are not excluded. The branching property can be summarised into the fact that given her type and birth time, the daughter process of any individual born is independent of all individuals not in her progeny (into which she herself is included).

We set out to prove that this branching property remains true for processes conditioned to die out. Initially, we shall not mention supercriticality, and only ask that the probability of extinction is non-zero for any starting type. (If that probability is one, the conditioning does not change anything!) Largely, the proof is a matter of conceptual clarity or discipline, which unfortunately forces us into the somewhat burdensome notation of probabilities on tree spaces, obscuring the essential simplicity of the matter.

The main idea behind the proof is, however, easily outlined. Indeed, consider an individual, and condition upon her progeny ultimately dying out. Her own life career is then affected precisely through her only being able to have daughters whose progeny in their turn must ultimately face extinction. In all other respects her life is independent of all others, once her type is given. This reestablishes the branching character, but with a suitably amended probability measure over her life career, which clearly is non-supercritical in the sense that the probability of ultimate extinction is one, from any starting type that can be realised.

If the original process is, furthermore, supercritical, *i.e.* has a positive Malthusian parameter, the conditioned process will turn out to be subcritical, in the sense of the Malthusian parameter being negative, if it exists.

## 2 Notation

### 2.1 The Ulam-Harris family space

We choose to work within the classical Ulam-Harris framework, [4], identifying individuals with sequences of natural numbers so that  $x = (x_1, x_2, \dots, x_n)$  denotes the  $x_n$ th child of the  $\dots$  of the  $x_2$ th child of the  $x_1$ th child of the ancestor. The ancestor is denoted by an “empty” sequence  $e$  (mnemonic for “empty” or “Eve”), and the set of all possible individuals is

$$\mathbb{T} = e \cup \bigcup_{n \in \mathbb{N}} \mathbb{N}^n.$$

The concatenation of  $x, y \in \mathbb{T}$  is  $xy$ , and thus  $ex = xe = x$  for all  $x \in \mathbb{T}$ .

For any  $e \neq x = (x_1, x_2, \dots, x_n)$   $x$ 's mother is  $mx = (x_1, \dots, x_{n-1})$ , her rank in the sibship is  $rx = x_n$ , and  $x$ 's generation  $g(x) = n$ . We agree that  $me = re = e$  and  $g(e) = 0$ . Hence,  $mrx = x$  for  $x \in \mathbb{T}$ , and  $m$  can be iterated so that  $m^n x$  is  $x$ 's  $n$ th grandmother, provided  $g(x) > n$ .

Clearly  $x$  stems from  $y$ , usually written  $x \succeq y$ , if  $m^n x = y$  for some  $n \in \mathbb{N} \cup \{0\}$ , or equivalently if there exists a  $z \in \mathbb{T} : x = yz$ . In this terminology,  $x$  stems from herself,  $x \preceq x$ . In other words,  $(\mathbb{T}, \preceq)$  is a partially ordered set (a semilattice). We define  $x \sim y$  if  $x \succeq y$  or  $x \preceq y$ , i.e.  $x$  and  $y$  are in direct line of descent. ( $\sim$  is not an equivalence relation.)

For  $A, B \subseteq \mathbb{T}$ ,  $x \in \mathbb{T}$ , we write  $x \succeq A$  if there exists a  $y \in A$  such that  $x \succeq y$ , and  $A \preceq B$  if  $x \succeq A$  for all  $x \in B$ . The progeny of  $A \subseteq \mathbb{T}$  is defined as  $\text{Pr } A = \{x \in \mathbb{T} : x \succeq A\}$ .

We call a set  $L \subset \mathbb{T}$  a stopping line, or *line* for short, if no two members of  $L$  are in direct line of descent:  $x, y \in L, x \neq y \Rightarrow x \not\sim y$ . We say that a line  $L$  is a *covering line* if for all  $x \in \mathbb{T}$  there exists a  $y \in L$  such that  $x \sim y$ .

## 2.2 Life space and population space

Let  $(\Omega_\ell, \mathcal{A}_\ell)$  be a *life space* so that  $\omega \in \Omega_\ell$  is a possible life career of individuals. Any individual property, such as mass at a certain age or life span, is viewed as a measurable function (with respect to the  $\sigma$ -algebra  $\mathcal{A}_\ell$ ) on the life space. This should be rich enough to support, at least, the functions  $\tau(k), \sigma(k)$  for  $k \in \mathbb{N}$ . Here  $\tau(k) : \Omega_\ell \rightarrow \mathbb{R}_+ \cup \{\infty\}$  is the mother's age at the  $k$ th child's birth,  $0 \leq \tau(1) \leq \tau(2) \leq \dots \leq \infty$ . If  $\tau(k) = \infty$ , then the  $k$ th child is never born.  $\sigma(k) : \Omega_\ell \rightarrow \mathcal{S}$  is the child's type, obtained at birth. The type space  $\mathcal{S}$  has a (countably generated)  $\sigma$ -algebra  $\mathcal{S}$ . The whole reproduction process is then the marked point process  $\xi$  with  $\xi(A \times B) = \#\{k : \sigma(k) \in A, \tau(k) \in B\}$  for  $A \in \mathcal{S}, B \in \mathcal{B}$ , the Borel algebra on  $\mathbb{R}_+$ .

The population space is defined as  $(\Omega, \mathcal{A}) = (S \times \Omega_\ell^\mathbb{T}, \mathcal{S} \times \mathcal{A}_\ell^\mathbb{T})$ .  $U_M$  is the projection  $\Omega \rightarrow \Omega_\ell^M$ , for  $M \subseteq \mathbb{T}$ . For simplicity  $U_x = U_{\{x\}}$  and similarly  $\text{Pr } x = \text{Pr}\{x\}$ . The following  $\sigma$ -algebras are important:

$$\mathcal{F}_L = \mathcal{S} \times \sigma(U_x : x \not\preceq L) = \mathcal{S} \times \sigma(U_x : x \notin \text{Pr } L),$$

for  $L \in \mathbb{T}$ . Since  $L \preceq M \Rightarrow \text{Pr } L \supseteq \text{Pr } M \Rightarrow \mathcal{F}_L \subseteq \mathcal{F}_M$ , it holds that  $(\mathcal{F}_L : L \in \mathbb{T})$  is a filtration under  $\preceq$ . In the usual manner, the definition of the  $\sigma$ -algebras  $\mathcal{F}_L$  can be extended to  $\sigma$ -algebras of events preceding random lines  $\mathcal{L}$  which are optional in the sense that events  $\{\mathcal{L} \preceq L\} \in \mathcal{F}_L$  [4].

Functions  $\xi, \tau(k)$  and  $\sigma(k)$  were defined on the life space but we want to be able to speak about these quantities pertaining to a given  $x \in \mathbb{T}$ . We write  $\xi_x = \xi \circ U_x$ ,  $x$ 's reproduction process,  $\tau_x = \tau(rx) \circ U_{mx}$ ,  $x$ 's mother's age at  $x$ 's birth, and  $x$ 's type  $\sigma_x = \sigma(rx) \circ U_{mx}$ . Note the difference between  $\tau(k)$  and  $\tau_k$ ,  $\sigma(k)$  and  $\sigma_k$ , for  $k \in \mathbb{N} \subset \mathbb{T}$ .

Finally, the process is anchored in real time by taking Eve to be born at time 0, and later birth times  $t_x, x \in \mathbb{T}$  recursively determined by  $t_e = 0$  and  $t_x = t_{mx} + \tau_x$  for  $e \neq x \in \mathbb{T}$ . The meaning of  $t_x = \infty$  is that  $x$  is never born, so that  $\mathcal{R} = \{x \in \mathbb{T} : t_x < \infty\}$  is the set of *realised* individuals. This set is optional,  $\mathcal{F}_{L \cap \mathcal{R}}$  is well defined [4], and so is the  $\sigma$ -algebra  $\mathcal{F}_{\mathcal{R}}$  of events pertaining only to realised individuals. The probability space restricted to such events is that where a branching processes really lives, cf. [6], [2].

### 2.3 The probability measure and branching property

The setup is that for each  $s \in \mathcal{S}$  there is a probability measure  $P(s, \cdot)$  on the life space  $(\Omega_\ell, \mathcal{A}_\ell)$ , such that the function  $s \rightarrow P(s, A)$  is measurable with  $A \in \mathcal{A}_\ell$ . For any  $s \in \mathcal{S}$  this kernel (the *life kernel*) defines a *population probability* measure  $\mathbb{P}_s$  on  $(\Omega, \mathcal{A})$  with an ancestor of type  $\sigma_e = s$  and such that given  $\sigma_x$ ,  $x$ 's life will follow the law  $P(\sigma_x, \cdot)$  independently of the rest of the process, see [4].

Indeed, the basic branching property of the whole process can be characterised by a generalisation of this in terms of the mappings  $S_x = (\sigma_x, U_{\text{Pr } x}) : \mathcal{S} \times \mathbb{T} \rightarrow \mathcal{S} \times \mathbb{T}$ , which renders  $x$  the new Eve. Let  $T_x = S_x^{-1}$  and  $\{A_x : x \in L\} \subseteq \mathcal{A}$ . Then,

$$\mathbb{P}_s \left( \bigcap_{x \in L} T_x A_x \middle| \mathcal{F}_L \right) = \prod_{x \in L} \mathbb{P}_{\sigma_x}(A_x)$$

for lines  $L$ . This remains true for optional lines and in particular

$$\mathbb{P}_s \left( \bigcap_{x \in L \cap \mathcal{R}} T_x A_x \middle| \mathcal{F}_{L \cap \mathcal{R}} \right) = \prod_{x \in L \cap \mathcal{R}} \mathbb{P}_{\sigma_x}(A_x), \quad (1)$$

where the intersection over the empty set is taken to be  $\Omega$  and the empty product is ascribed the value one. The interpretation is that the daughter processes of all realised individuals  $x$  in a line are independent given the prehistory of the line with the population probability measure  $\mathbb{P}_{\sigma_x}$ , the only dependence upon the past thus being channelled through the type  $\sigma_x$  and the placing in time  $t_x$ . This is the branching property. We shall see that it remains true for processes, bound to die out.

## 3 Conditioning on extinction

Denote by  $E$  the event that the branching process starting from Eve dies out, i.e. that  $\mathcal{R}$  has only a finite number of elements. Let  $q_s = \mathbb{P}_s(E)$  and  $E_x = T_x E$ . Write  $\tilde{\mathbb{P}}_s(\cdot) = \mathbb{P}_s(\cdot | E)$ , which clearly only makes sense for  $s \in \mathcal{S}$  such that  $q_s > 0$ , and let  $\mathbb{E}_s$  denote expectation with respect to  $\tilde{\mathbb{P}}_s$ .

**Theorem 1.** *Any branching process conditioned on extinction remains a branching process, but with extinction probability one. Its life kernel is  $\tilde{P}(s, A) := \tilde{\mathbb{P}}_s(\mathcal{S} \times A \times \Omega_\ell^{\mathbb{T} \setminus \{e\}})$  for  $A \in \mathcal{A}_\ell$ . Thus, for any covering lines  $L$  and  $\{A_x : x \in L\} \subseteq \mathcal{A}$*

$$\tilde{\mathbb{P}}_s \left( \bigcap_{x \in L \cap \mathcal{R}} T_x A_x \middle| \mathcal{F}_{L \cap \mathcal{R}} \right) = \prod_{x \in L \cap \mathcal{R}} \tilde{\mathbb{P}}_{\sigma_x}(A_x). \quad (2)$$

Furthermore, the Radon-Nikodym derivative  $d\tilde{\mathbb{P}}_s/d\mathbb{P}_s$  with respect to the  $\sigma$ -algebra  $\mathcal{F}_{L \cap \mathcal{R}}$  is given by

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_{L \cap \mathcal{R}}} = \frac{1}{q_s} \prod_{x \in L \cap \mathcal{R}} q_{\sigma_x}. \quad (3)$$

*Proof.* First, note that

$$E = \bigcap_{x \in L \cap \mathcal{R}} T_x E,$$

for covering lines  $L$ . Indeed, since  $\{L \cap \mathcal{R} = \emptyset\} \subseteq E$  and intersection over an empty index set yields the full space,

$$\begin{aligned} E &= (E \cap \{L \cap \mathcal{R} = \emptyset\}) \cup (E \cap \{L \cap \mathcal{R} \neq \emptyset\}) \\ &= \left( \{L \cap \mathcal{R} = \emptyset\} \cap \bigcap_{x \in L \cap \mathcal{R}} T_x E \right) \cup \left( \{L \cap \mathcal{R} \neq \emptyset\} \cap \bigcap_{x \in L \cap \mathcal{R}} T_x E \right) \\ &= \bigcap_{x \in L \cap \mathcal{R}} T_x E. \end{aligned}$$

The branching property (1) implies that

$$\mathbb{P}_s(E|\mathcal{F}_L) = \prod_{x \in L \cap \mathcal{R}} q_{\sigma_x} = \mathbb{P}_s(E|\mathcal{F}_{L \cap \mathcal{R}}). \quad (4)$$

Hence, for any covering line  $L$  and  $A \in \mathcal{F}_L$

$$\tilde{\mathbb{P}}_s(A) = \mathbb{E}_s \left[ \mathbb{P}_s(E|\mathcal{F}_L); A \right] / q_s = \mathbb{E}_s \left[ \prod_{x \in L \cap \mathcal{R}} q_{\sigma_x}; A \right] / q_s,$$

and thus (3) holds. Equations (1) and (4) yield

$$\mathbb{P}_s \left( \bigcap_{x \in L \cap \mathcal{R}} T_x A_x \middle| \mathcal{F}_{L \cap \mathcal{R}} \right) = \prod_{x \in L \cap \mathcal{R}} q_{\sigma_x} \tilde{\mathbb{P}}_{\sigma_x}(A_x) = \mathbb{P}_s(E|\mathcal{F}_{L \cap \mathcal{R}}) \prod_{x \in L \cap \mathcal{R}} \tilde{\mathbb{P}}_{\sigma_x}(A_x),$$

and (2) follows.  $\square$

*Remark.* For  $L = \mathbb{N}$ , the first generation,

$$\left. \frac{d\tilde{\mathbb{P}}_s}{d\mathbb{P}_s} \right|_{\mathcal{F}_{\mathbb{N} \cap \mathcal{R}}} = \frac{1}{q_s} \prod_{k \in \mathbb{N}: k \leq X} q_{\sigma_k},$$

where  $X := \xi_e(\mathcal{S} \times \mathbb{R}_+)$  is Eve's total offspring. In single-type processes with extinction probability  $q$  therefore

$$\left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_{\mathbb{N} \cap \mathcal{R}}} = q^{X-1} \quad (5)$$

and in the Bienaymé-Galton-Watson case

$$\tilde{\mathbb{P}}(X = k) = \mathbb{E}[q^{X-1}; X = k] = \mathbb{P}(X = k)q^{k-1},$$

in perfect agreement with [1, Theorem I.12.3].

*Example.* Consider single-type Sevastyanov splitting processes, where individuals have a life span distribution  $G$  and at death split into  $k$  particles with probability  $p_k(u)$ , if the life span  $L = u$ . By (5) we conclude that

$$\tilde{\mathbb{P}}(X = k, L \in du) = \mathbb{E}[q^{X-1}; X = k, L \in du] = p_k(u)q^{k-1}G(du).$$

Hence

$$\tilde{G}(du) = \tilde{P}(L \in du) = \sum_{k=0}^{\infty} p_k(u)q^{k-1}G(du)$$

and  $\tilde{\mathbb{P}}(X = k, L \in du) = \tilde{p}_k(u)\tilde{G}(du)$ , with

$$\tilde{p}_k(u) = \frac{p_k(u)q^{k-1}}{\sum_{i=0}^{\infty} p_i(u)q^{i-1}},$$

and the conditioned Sevastyanov process remains a Sevastyanov process with life span distribution  $\tilde{G}$  and splitting probabilities  $\tilde{p}_k(u)$ .

Finally, we address the question of whether a supercritical process conditioned on extinction is subcritical. It is clear that the conditioned branching process has extinction probability one, for any starting type, but this would also be the case if the process were nontrivially critical.

In the single type case it follows from (5) that the expected total number of children per individual in the conditioned process satisfies

$$\tilde{\mathbb{E}}[X] = \mathbb{E}[Xq^{X-1}] = f'(q),$$

in terms of the offspring generating function  $f$  of the embedded Bienaymé-Galton-Watson process. It is well known that  $f'(q) < 1$  if the original process was supercritical, see [1], and we conclude that the conditioned process is subcritical.

Also in the general case it is enough to consider the embedded generation counting process. If  $X_n(A) = \text{card}\{x \in \mathbb{N}^n \cap \mathcal{R} : \sigma_x \in A\}$  denotes the number of individuals in  $n$ th generation of type  $A$  and  $q := \sup_{\mathcal{S}} q_s < 1$ ,

$$\tilde{\mathbb{E}}_s[X_n(\mathcal{S})] = \mathbb{E}_s\left[X_n(\mathcal{S})e^{\int_{\mathcal{S}} \log q_r X_n(dr)}\right]/q_s \leq \mathbb{E}_s\left[X_n(\mathcal{S})q^{X_n(\mathcal{S})}\right]/q_s \rightarrow 0,$$

as  $n \rightarrow \infty$  (by dominated convergence, since  $X_n(\mathcal{S})$  must either tend to zero or to infinity). But the expected size of the embedded process tending to zero means exactly that the process is subcritical.

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